يتألف الإختبار من 4 أسئلة في صفحتين. برجاء بدء إجابة كل فرع من إحدى نهايتي ورقة الإجابة.

Question (1) [35 points]

(a) Evaluate the integral $\int_0^1 \sqrt[4]{\frac{1}{x}-1}\,dx$. **[5 points]**

(b) Find the series solution of $(x^2+1)y''-4xy=0$ about $x_0=0$. **[6 points]**

(c) Find the Fourier integral representation of the function **[6 points]**

$$f(x)=\begin{cases} e^{-x}, & 0\le x \\ 0, & x<0 \end{cases}$$

Compute the values of that Fourier integral when substitute $x=1$, $x=0$, and $x=-1$.

(d) Based on the orthogonality of Legendre polynomials of first kind, deduce the value of A_n to expand a function $f(x)$ in the series form $f(x)=\sum_{n=1}^{\infty} A_n P_n(x)$. **[6 points]**

(e) Evaluate $\frac{d}{dx}(J_2(x)J_3(x)J_5(x))$ in a form that contains $J_6(x)$. **[6 points]**

(f) The generating function of polynomials $L_n(x)$ is $\frac{1}{1-t}e^{(\frac{-xt}{1-t})}=\sum_{n=0}^{\infty} t^n L_n(x)$. Show that $L_n(0)=1$ for all n. **[6 points]**

Question (2) يسمح بالإجابة باللغة العربية عند الحاجة للشرح [20 points]

(a) What is the basic principle of the method of separation of variables for solving PDEs? **[3 points]**

(b) Deduce the wave equation modeling the vibration of finite string with fixed ends. Write the physical assumptions of the model. **[7 points]**

(c) Explain how the solution of the standard heat equation by separation of variables differs for the case of finite rod from the case of semi-infinte rod. **[5 points]**

(d) Which of the Fourier representations is suitable for $f(x)=\tan(x)$: Fourier trigonometric series, Fourier half-range expansion, or Fourier integral? Why? **[5 points]**

3. (a) [10 pts] Discuss the mapping $w = e^z$.

(b) [10 pts] Expand the function $f(z) = \frac{z^2-2z+5}{(z-2)(z^2+1)}$ in the ring $1 < |z| < 2$.

(c) [10 pts] Find all roots of $\sqrt{\cos\theta - i\sin\theta}$. Determine the values of θ for which these roots are

i. pure real,

ii. pure imaginary.

4. (a) [10 pts] Evaluate the integral $\int_C \frac{z}{\bar{z}}\,dz$, where c is the boundary of the half ring $1 \le |z| \le 2$, $\operatorname{Im} z \ge 0$.

(b) [10 pts] Show that u is harmonic, find a harmonic conjugate v and express $f = u + iv$ as a function of z if $u = -\frac{y}{x^2+y^2}$.

(c) [10 pts] Use the residue theorem to evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2(x^2+b^2)}$, $a > 0,\ b > 0$.

I. A. El-Awadi